Ray tracing and generation of entangled light – Marking Scheme

Remark: When student's solutions are correct and also show how solutions were obtained, it gets full credit. Only when student's solutions are incorrect or partially correct, the followings apply.

Part A. Light propagation in isotropic dielectric media

A.1 0.4 pt Solution: $\frac{1}{\sqrt{\mu_0\epsilon}}$	Realize that the phase velocity is given by $\frac{\omega}{k}$	0.2 pt
	Correct expression for $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon}}$	0.2 pt
A.2 0.2 pt Solution: $n = c\sqrt{\mu_0\epsilon}$	Correct relation $\frac{\omega}{k} = \frac{c}{n}$	0.1 pt
	Correct expression $n = c\sqrt{\mu_0\epsilon}$	0.1 pt
A.3 0.4 pt Solution: $\hat{S} = \hat{k}$ $v_r = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$	Correct expression for direction of $\hat{S}$	0.2 pt
	Correct result of computing the ratio S/u	0.1 pt
	Correct expression for $v_r = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$	0.1 pt

Part B. Light propagation in uniaxial dielectric media

B.1 1.5pt Solution: $n = n_o$ $\hat{B} = \pm\hat{k}\times\hat{y} =$ $\pm(-\cos\theta, 0\sin\theta)$ $\hat{D} = \pm\hat{y}$ $n = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$ $\hat{B} = \pm\hat{y}$ $\hat{D} = \pm\hat{y}\times\hat{k} =$ $\pm(\cos\theta, 0, -\sin\theta)$ $\theta = 0$ *or* π only one refractive index is allowed.	Realize that the determinant associated with equations for electric field has to vanish and correctly write out the form of the determinant.	0.2 pt
	Correct equation for n	0.1 pt
	Correct expressions for n n_o: 0.1pt, $\frac{n_o n_e}{\sqrt{n_e^2\sin^2\theta + n_o^2\cos^2\theta}}$: 0.2pt	0.3 pt
	Correct expressions for $\hat{B}$ (each direction is 0.2 pt, both + and – are given the fill credit)	0.4 pt

	Correct expressions for $\widehat{D}$ (each direction is 0.2 pt, both + and – are given the fill credit)	0.4 pt
	Correct value for the angle with only one refractive index	0.1 pt
B.2 0.8pt Solution: $n = n_o$ $\hat{E} = \pm\hat{y}$ ordinary light ray $\tan\alpha = 0$ $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$ $\hat{E} = \pm\frac{1}{\sqrt{n_e^4 cos^2\theta + n_o^4 \sin^2\theta}}$ $(-n_e^2\cos\theta, 0, n_o^2\sin\theta)$ Extraordinary light ray $\tan\alpha = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2\tan^2\theta}$	Correct ratio of $E_z\colon E_x$ for the case of $n = \frac{n_o n_e}{\sqrt{n_e^2 \sin^2\theta + n_o^2 \cos^2\theta}}$: 0.1 pt Correct expression for the polarization of the corresponding refractive index: $\hat{E} = \pm\hat{y}$: 0.1 pt $\hat{E} = \pm\frac{1}{\sqrt{n_e^4 cos^2\theta + n_o^4 \sin^2\theta}}$ $(-n_e^2\cos\theta, 0, n_o^2\sin\theta)$: 0.1pt	0.3 pt
	Correct expressions for the angle of $\vec{E}$ and $\vec{D}$ relative to x axis: 0.1 pt Correct expression for $\tan\alpha$: $\tan\alpha = 0$: 0.1 pt $\tan\alpha = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2\tan^2\theta}$: 0.1 pt	0.3 pt
	Correctly indicate types of light rays: Ordinary light ray 0.1 pt Extraordinary light ray 0.1pt	0.2 pt
B.3 0.6pt Solution: $n = n_o$ $\hat{E} = \pm\hat{z}\times\hat{k} / \sin\theta$ ordinary light ray $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$ $\hat{E} = \pm\frac{1}{\sqrt{n_e^4 cos^2\theta + n_o^4 \sin^2\theta}}$ $\frac{-n_e^2\cos^2\theta\hat{k} + (n_e^2\sin^2\theta + n_o^2\cos^2\theta)\hat{z}}{\sin\theta}$	Realize that the axial symmetry and replace $\hat{x}$ by $\hat{k}_\perp$	0.2pt
	Correct expressions for the polarization of the corresponding refractive index: $\hat{E} = \pm\hat{y}$: 0.1 pt $\hat{E} = \pm\frac{1}{\sqrt{n_e^4 cos^2\theta + n_o^4 \sin^2\theta}}\times$ $\frac{-n_e^2\cos^2\theta\hat{k} + (n_e^2\sin^2\theta + n_o^2\cos^2\theta)\hat{z}}{\sin\theta}$: 0.1pt	0.2pt

extraordinary light ray	Correct expressions for n (0.1pt) and indications for type of light rays (0.1pt)	0.2pt
B.4 0.8 pt Solution: $n = n_o$: $\tan \alpha_r = 0$ $v_r = \frac{c}{n_o}$	Correct expressions for $\tan \alpha_r$ (each expression 0.1pt for different n)	0.2 pt
$\hat{S} = (\sin\theta, 0, \cos\theta)$ $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$:	Correct expressions for v_r (each expression 0.1pt for different n)	0.2 pt
$\tan \alpha_r = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2 \tan^2\theta} = \tan\alpha$ $v_r = \frac{c}{n_o n_e} \frac{\sqrt{n_e^4 cos^2\theta + n_o^4 \sin^2\theta}}{\sqrt{n_e^2 cos^2\theta + n_o^2 \sin^2\theta}}$	Correct expressions for $\hat{S}$ (each expression 0.1pt for different n)	0.2 pt
$\hat{S} = \frac{1}{\sqrt{n_e^4 cos^2\theta + n_o^4 \sin^2\theta}} \times$ $(n_o^2 \sin\theta, 0, n_e^2 \cos\theta)$ $n_s = \sqrt{(\hat{S}\cdot\hat{x})^2 n_e^2 + (\hat{S}\cdot\hat{z})^2 n_o^2}$	Correct expression for n_s	0.2 pt
B.5 1.1 pt Solution: $\bar{A} = P_1(n^2 \sin^2\theta_1 - P_1)$ $\bar{B} = -2P_3(n^2 \sin^2\theta_1 - P_1)$ $\bar{C} = P_2 n^2 \sin^2\theta_1 - P_3^2$	Indicate that the path is determined by the optical path length $d_1 n_{s_1} + d_2 n_{s_2}$ where d_1 and d_2 are distances ,connecting A to O and O to B (0.1 pt). n_{s_1} and n_{s_2} are the corresponding refractive indices of the path d_1 and d_2 (0.2 pt)	0.3 pt
$\phi = 0$: $\tan\theta_2 = \frac{n n_e \sin\theta_1}{n_o\sqrt{n_o^2 - n^2 \sin^2\theta_1}}$	Correct expression for the optical path length in terms of geometric factors (such as θ_1, ϕ, θ_2 and coordinates of points A and B) Each minor error in expression: -0.1 pt	0.3 pt
$\phi = \pi/2$: $\tan\theta_2 = \frac{n n_o \sin\theta_1}{n_o\sqrt{n_o^2 - n^2 \sin^2\theta_1}}$	Correct expression for $\bar{A}$	0.1 pt
	Correct expression for $\bar{B}$	0.1 pt

	Correct expression for $\bar{C}$	0.1 pt
	Correct expression for $\tan\theta_2$ when $\phi = 0$	0.1 pt
	Correct expression for $\tan\theta_2$ when $\phi = \pi/2$	0.1 pt

Part C. Entanglement of light

C.1 0.8 pt Solution: Relations: $\omega = \omega_1 \pm \omega_2$ $\vec{k} = \vec{k}_1 \pm \vec{k}_2$ $\vec{k} = \vec{k}_1 \pm \vec{k}_2$: momentum conservation $\omega = \omega_1 \pm \omega_2$: energy conservation Splitting: $\omega = \omega_1 + \omega_2$ $\vec{k} = \vec{k}_1 + \vec{k}_2$	Correct expressions for $\omega = \omega_1 \pm \omega_2$ (+: 0.1pt, −: 0.1pt)	0.2 pt
	Correct expressions for $\vec{k} = \vec{k}_1 \pm \vec{k}_2$ (+: 0.1pt, −: 0.1pt)	0.2 pt
	Adding $\hbar$ and interpretate $\hbar\vec{k} = \hbar\vec{k}_1 \pm \hbar\vec{k}_2$ as momentum conservation	0.1 pt
	Adding $\hbar$ and interpretate $\hbar\omega = \hbar\omega_1 \pm \hbar\omega_2$ as energy conservation	0.1 pt
	Correct expressions for splitting of $\omega = \omega_1 + \omega_2$ (0.1pt) and $\vec{k} = \vec{k}_1 + \vec{k}_2$ (0.1pt)	0.2 pt
C.2 0.8 pt Solution: $o \to o + o$ $e \to e + e$	Indicating that there is a confliction for splitting into the same type of the light ray due to that the refractive indices n_o and n_e are both increasing functions of ω.	0.4 pt
	Correctly listing $o \to o + o$	0.2 pt
	Correctly listing e→ $e + e$	0.2 pt
	Extra listing of splitting: -0.2 pt for each listing	

C.3 1.3 pt Solution: $M = \frac{K_o(1-N_e(\Omega_e,\theta)\cot\theta)+K_e}{2K_oK_e}$ $N = -\frac{N_e}{2M}$ $L = -(\Omega-\Omega_e)\left(\frac{1}{u_e}-\frac{1}{u_o}\right)+\frac{N_e^2}{4M}$ Angle between the axis of the cone and z' is $\frac{N}{K_o}$ $(= -\frac{K_eN_e}{K_o(1-N_e(\Omega_e,\theta)\cot\theta)+K_e})$ Angle of the cone is $\frac{\sqrt{L/M}}{K_o}$ $(= -\frac{(\Omega-\Omega_e)}{MK_o}\left(\frac{1}{u_e}-\frac{1}{u_o}\right)+\frac{N_e^2}{4M^2K_o})$	Realize the conservation of momentum along z direction: $K_p = k_{1z} + k_{2z}$	0.1 pt
	Correct expansion of k_{2z} Minor errors for numerical factors: -0.1 pt	0.3 pt
	Correct expansion of k_{1z} in frequency Minor errors for numerical factors: -0.1 pt	0.2 pt
	Correct expansion of k_{1z} in momentum Minor errors for numerical factors: -0.1 pt	0.2 pt
	Correct expression for M	0.1 pt
	Correct expression for N	0.1 pt
	Correct expression for L	0.1 pt
	Correct expression for the angle between the axis of the cone and z' (using N and K_o)	0.1 pt
	Correct expression for the angle of the cone (using L, M and K_o)	0.1pt
C.4 0.9 pt Solution: $P(\alpha,\beta) = \frac{1}{2}\sin^2(\alpha+\beta)$ $P(\alpha,\beta_\perp) = \frac{1}{2}\cos^2(\alpha+\beta)$ $P(\alpha_\perp,\beta) = \frac{1}{2}\sin^2(\alpha+\beta)$ $P(\alpha_\perp,\beta_\perp) = \frac{1}{2}\cos^2(\alpha+\beta)$	Correctly expressing the electric fields along $\hat{x}'$ and $\hat{y}'$ direction in terms of the electric fields along the direction of the polarizer and perpendicular to the direction of polarizer for individual a-photon (0.1pt) and b-photon (0.1pt)	0.2 pt
	Correctly expressing the entangled photon pair state $\frac{1}{\sqrt{2}}(\lvert\hat{x}'_a\rangle\lvert\hat{y}'_b\rangle + \lvert\hat{y}'_a\rangle\lvert\hat{x}'_b\rangle)$ in terms of combination of states using directions of	0.3 pt

	the polarizer: $\lvert\alpha_x\rangle\lvert\beta_x\rangle - \lvert\alpha_y\rangle\lvert\beta_y\rangle$, $\lvert\alpha_x\rangle\lvert\beta_y\rangle - \lvert\alpha_y\rangle\lvert\beta_x\rangle$	
	Correct expression of $P(\alpha,\beta)$	0.1 pt
	Correct expression of $P(\alpha,\beta_\perp)$	0.1 pt
	Correct expression of $P(\alpha_\perp,\beta)$	0.1 pt
	Correct expression of $P(\alpha_\perp,\beta_\perp)$	0.1 pt
C.5 0.5 pt Solution: $S = \lvert\cos 2(\alpha-\beta) - \cos 2(\alpha-\beta')\rvert + \lvert\cos 2(\alpha'-\beta) + \cos 2(\alpha'-\beta')\rvert$ Value of $S = 2\sqrt{2} > 2$ Inconsistent with classical theories	Correct expression of $E(\alpha,\beta)$ in terms of $P(\alpha,\beta_\perp)$, $P(\alpha_\perp,\beta)$, $P(\alpha_\perp,\beta)$ and $P(\alpha_\perp,\beta_\perp)$	0.3 pt
	Correct expression of $E(\alpha,\beta)$ in terms of α and β	0.1 pt
	Correct value of S and consistency with classical theories.	0.1 pt